

Multisymmetric polynomials in dimension three

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Abstract

The polarizations of one relation of degree five and two relations of degree six minimally generate the ideal of relations among a minimal generating system of the algebra of multisymmetric polynomials in an arbitrary number of three-dimensional vector variables. In the general case of n -dimensional vector variables, a relation of degree $2n$ among the polarized power sums is presented such that it is not contained in the ideal generated by lower degree relations.

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1 Introduction

The symmetric group S_n acts on the n -dimensional complex vector space $V := \mathbb{C}^n$ by permuting coordinates. Consider the diagonal action of S_n on the space $V^m := V \oplus \cdots \oplus V$ of m -tuples of vectors from V . The *algebra of multisymmetric polynomials* is the corresponding ring of invariants $R_{n,m} := \mathbb{C}[V^m]^{S_n}$, consisting of the polynomial functions on V^m that are constant along the S_n -orbits.

In the special case $m = 1$, $R_{n,1}$ is a polynomial ring generated by the elementary symmetric polynomials (or by the first n power sums). It is classically known (see [14], [13], [16]) that the polarizations of the elementary symmetric polynomials constitute a minimal \mathbb{C} -algebra generating system of $R_{n,m}$ for an arbitrary m . The ideal of relations among these generators is not completely understood, although it was classically studied in [9], [10], [11], and in a couple of more recent papers (see the references in Section 2).

An explicit finite presentation of $R_{n,m}$ by generators and relations is known, see Proposition 2.1 below. Note that the price for having a uniform description of the ideal of relations in Proposition 2.1 is the inclusion of redundant elements in the system of generators.

In the present paper for $n = 3$ we determine a minimal system of generators of the ideal of relations among a minimal generating system of $R_{3,m}$. We exploit the natural action of the general

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linear group GL_m on $R_{3,m}$. Identify V^m with the space $\mathbb{C}^{n \times m}$ of $n \times m$ matrices. The complex general linear group GL_m acts on $\mathbb{C}^{n \times m}$ by matrix multiplication from the right: $x \mapsto xg^{-1}$ ($x \in \mathbb{C}^{n \times m}$, $g \in \mathrm{GL}_m$). As usual, this induces an action of GL_m on the coordinate ring $\mathbb{C}[V^m]$ by linear substitution of variables. Since the action of GL_m on V^m commutes with the action of S_n , the algebra $R_{n,m}$ is a GL_m -submodule in the coordinate ring $\mathbb{C}[V^m]$. In particular, we may choose a minimal system of homogeneous \mathbb{C} -algebra generators of $R_{n,m}$ whose \mathbb{C} -linear span is a GL_m -submodule $W_{n,m}$ in $\mathbb{C}[V^m]$. Write $S(W_{n,m})$ for the symmetric tensor algebra of $W_{n,m}$. (This is polynomial ring, with one variable associated to each element of a fixed basis of $W_{n,m}$.) Endow the algebra $S(W_{n,m})$ with the GL_m -module structure induced by the representation of GL_m on $W_{n,m}$. Consider the natural \mathbb{C} -algebra surjection $\varphi : S(W_{n,m}) \rightarrow R_{n,m}$ extending the identity map on $W_{n,m}$. The ideal $\ker(\varphi)$ of relations among the chosen generators of $R_{n,m}$ is a GL_m -submodule of $S(W_{n,m})$.

The coordinate ring $\mathbb{C}[V^m] = \mathbb{C}[x_{ij} \mid i = 1, \dots, n; j = 1, \dots, m]$ is an nm -variable polynomial algebra, where x_{ij} stands for the i th coordinate function on the j th vector component. Given a monomial $w = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ in the m -variable polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$, set

$$[w] := \sum_{i=1}^n x_{i1}^{\alpha_1} \cdots x_{im}^{\alpha_m}.$$

These elements of $R_{n,m}$ are called the *polarized power sums*, and

$$\{[x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \mid \sum_{j=1}^m \alpha_j \leq n\} \quad (1)$$

is a minimal system of \mathbb{C} -algebra generators of $R_{n,m}$.

Denote by $W_{n,m}$ the subspace of $R_{n,m}$ spanned by the set (1). This is a GL_m -submodule. In the case $n = 3$, we present three explicit elements in the kernel of the surjection $\varphi : S(W_{3,m}) \rightarrow R_{3,m}$ such that each of them generates an irreducible GL_m -submodule in $S(W_{3,m})$, and the union of any \mathbb{C} -bases of these three irreducible GL_m -submodules constitutes a minimal generating system of the ideal $\ker(\varphi)$ (see Theorem 3.1).

For arbitrary n we point out a connection between multisymmetric polynomials and vector invariants of the full orthogonal group, and use this to show that a homogeneous system of generators of the ideal of relations between the polarized power sums must contain a relation of degree $2n$ (see Theorem 3.2 for the precise statement).

2 Preliminaries

Denote by \mathcal{M}_m the set of monomials in the polynomial algebra $\mathbb{C}[x_1, \dots, x_m]$, and for a natural number d denote by \mathcal{M}_m^d the subset of monomials of degree at most d . To each $w \in \mathcal{M}_m$ associate an indeterminate $t(w)$, and take the commutative polynomial algebra $\mathcal{F}_{n,m} := \mathbb{C}[t(w) \mid w \in \mathcal{M}_m]$ in infinitely many variables. For each $d \in \mathbb{N}$ it contains the subalgebra $\mathcal{F}_{n,m}^d := \mathbb{C}[t(w) \mid w \in \mathcal{M}_m^d]$. In particular, we identify $\mathcal{F}_{n,m}^n$ and $S(W_{n,m})$ in the obvious way: by definition, $\{[w] \mid w \in \mathcal{M}_m^n\}$ is a \mathbb{C} -vector space basis of $W_{n,m}$, and the map $[w] \mapsto t(w)$ extends uniquely to a \mathbb{C} -algebra isomorphism $S(W_{n,m}) \cong \mathcal{F}_{n,m}^n$. We denote by $\varphi_{n,m}$ the surjection $\varphi_{n,m} : \mathcal{F}_{n,m} \rightarrow R_{n,m}$ given by $\varphi(t(w)) = [w]$ for all $w \in \mathcal{M}_m$. The restriction of $\varphi_{n,m}$ to $\mathcal{F}_{n,m}^d$ will be denoted by $\varphi_{n,m}^d$; it is a surjection onto $R_{n,m}$ whenever $d \geq n$. To simplify notation later in the text, we sometimes write \mathcal{F} instead of $\mathcal{F}_{n,m}^n$ and φ instead of $\varphi_{n,m}^n$.

We recall some elements in the kernel of $\varphi_{n,m}$. By a *distribution* of the set $\{1, \dots, n+1\}$ we mean a set $\pi := \{\pi_1, \dots, \pi_h\}$ of pairwise disjoint non-empty subsets whose union is $\bigcup_{i=1}^h \pi_i = \{1, \dots, n+1\}$. Write D_{n+1} for the set of distributions of $\{1, \dots, n+1\}$. Take monomials $w_1, \dots, w_{n+1} \in \mathcal{M}_m$, and set

$$\Psi_{n+1}(w_1, \dots, w_{n+1}) = \sum_{\pi \in D_{n+1}} \prod_{\pi_i \in \pi} (-1)^{(|\pi_i| - 1)!} \cdot t\left(\prod_{s \in \pi_i} w_s\right). \quad (2)$$

Proposition 2.1 *The kernel of the surjection $\varphi_{n,m}^{n^2-n+2} : \mathcal{F}_{n,m}^{n^2-n+2} \rightarrow R_{n,m}$ is generated as an ideal by the elements $\Psi_{n+1}(w_1, \dots, w_{n+1})$, ranging over all choices of monomials $w_i \in \mathcal{M}_m$ with $\deg(w_1 \cdots w_{n+1}) \leq n^2 - n + 2$.*

Proposition 2.1 is a special case of a result in [6] dealing with vector invariants of a class of complex reflection groups. In loc. cit. we first gave a simple short proof of an infinite version about the kernel of $\mathcal{F}_{n,m} \rightarrow R_{n,m}$ (see also [2], [3], [4], [15] for related work). Then we applied Derksen's general degree bound for syzygies from [5] and ideas of Garsia and Wallach from [8] to derive in particular the above finite presentation of $R_{n,m}$ in [6].

To produce elements in $\ker(\varphi) = \ker(\varphi_{n,m}^n) = \ker(\varphi_{n,m}) \cap \mathcal{F}_{n,m}^n$ we shall start with the relations in Proposition 2.1 belonging to $\ker(\varphi_{n,m})$ and eliminate the variables $t(w)$ with $\deg(w) > n$. There is one exception, we construct an element J in $\ker(\varphi_{n,n}^2) = \ker(\varphi) \cap \mathcal{F}_{n,n}^2$ by another method as follows:

$$J := \det \begin{pmatrix} t(x_1^2) & t(x_1 x_2) & \dots & t(x_1 x_n) & t(x_1) \\ t(x_2 x_1) & t(x_2^2) & \dots & t(x_2 x_n) & t(x_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t(x_n x_1) & t(x_n x_2) & \dots & t(x_n^2) & t(x_n) \\ t(x_1) & t(x_2) & \dots & t(x_n) & n \end{pmatrix}. \quad (3)$$

Proposition 2.2 *The element J belongs to $\ker(\varphi_{n,n}^2)$, and $g \cdot J = \det^2(g)J$ for any $g \in \mathrm{GL}_n$.*

Proof. Applying φ to the entries of the $(n+1) \times (n+1)$ matrix in (3) we get the matrix $X^T \cdot X$, where

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} & 1 \\ x_{21} & x_{22} & \dots & x_{2n} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} & 1 \end{pmatrix},$$

and X^T denotes the transpose of X . Since X has size $n \times (n+1)$, the rank of $X^T X$ is at most n , hence $\det(X^T X) = 0$, showing that $J \in \ker(\varphi)$.

To explain the second statement, let us describe the GL_m -action on $\mathcal{F} = S(W_{n,m})$ more explicitly. First of all, GL_m acts by \mathbb{C} -algebra automorphisms on the m -variable polynomial ring $\mathbb{C}[x_1, \dots, x_m]$. Namely, $g \in \mathrm{GL}_m$ sends the variable x_i to the i th entry of the row vector $(x_1, x_2, \dots, x_m) \cdot g$ (matrix multiplication). Note that the degree d homogeneous component of $\mathbb{C}[x_1, \dots, x_m]$ is GL_m -stable, and as a GL_m -module, it is isomorphic to the d th symmetric tensor power of the natural m -dimensional representation of GL_m on the space \mathbb{C}^m of column vectors. Write U for the \mathbb{C} -linear span of \mathcal{M}_m^n in $\mathbb{C}[x_1, \dots, x_m]$ (so U is the sum of the homogeneous components of degree $1, 2, \dots, n$). It is easy to see that the \mathbb{C} -linear map from $U \rightarrow R_{n,m}$ induced by $w \mapsto [w]$ ($w \in \mathcal{M}_m^n$) is a GL_m -module isomorphism, so we have $U \cong W_{n,m}$ as a GL_m -module. This shows that the effect of $g \in \mathrm{GL}_m$ on a variable $t(w)$ of $\mathcal{F}_{n,m}^n$ is given by the

formula $g \cdot t(w) = t(g \cdot w)$, where for an arbitrary polynomial $f = \sum_{w \in \mathcal{M}_m} a_w w \in \mathbb{C}[x_1, \dots, x_m]$, we shall mean by $t(f)$ the element $\sum_{w \in \mathcal{M}_m} a_w t(w) \in \mathcal{F}_{n,m}$.

These considerations show that applying $g \in \mathrm{GL}_n$ to all entries of the matrix Y in (3) we get the matrix $(\tilde{g})^T Y \tilde{g}$, where \tilde{g} stands for the $(n+1) \times (n+1)$ block diagonal matrix $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Therefore our statement follows by multiplicativity of the determinant. \square

The idea of applying the GL_m -module structure in the study of generators and relations of rings of invariants $\mathbb{C}[V^m]^G$ (where G is a group of linear transformations on V) is well known, see for example section 5.2.7 in [7], or [1] for a recent application. We collect some necessary facts on representations of GL_m on polynomial rings.

The representation of GL_m on $S(W_{n,m})$ is a *polynomial representation* (cf. [12]). Recall that polynomial GL_m -modules are completely reducible. The isomorphism classes of irreducible polynomial representations are labeled by the set Par_m of partitions with at most m non-zero parts. By a partition $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathrm{Par}_m$ we mean a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0$ of non-negative integers, and write $h(\lambda)$ for the number of non-zero parts of λ . Given $\lambda \in \mathrm{Par}_m$ we denote by V_λ a copy of an irreducible polynomial GL_m -module labeled by λ . For example, $V_{(k)}$ is isomorphic to the degree k homogeneous component of $\mathbb{C}[x_1, \dots, x_m]$, the k th symmetric tensor power of the natural GL_m -module \mathbb{C}^m , and $V_{(1^k)} := V_{(1, \dots, 1)} \cong \bigwedge^k(\mathbb{C}^m)$, the k th exterior power of \mathbb{C}^m . We have the GL_m -module isomorphism

$$W_{n,m} \cong V_{(1)} \oplus V_{(2)} \oplus \dots \oplus V_{(n)}.$$

Write U_m for the subgroup of unipotent upper triangular matrices in GL_m , and write $\mathbb{T} \cong (\mathbb{C}^\times)^m = \mathbb{C}^\times \times \dots \times \mathbb{C}^\times$ for the maximal torus consisting of diagonal matrices. Given a polynomial GL_m -module M , we say that a non-zero $v \in M$ is a *highest weight vector of weight λ* if v is fixed by U_m , \mathbb{T} stabilizes $\mathbb{C}v$ and acts on it by the weight λ , i.e. $\mathrm{diag}(z_1, \dots, z_m) \cdot v = (z_1^{\lambda_1} \dots z_m^{\lambda_m})v$. In this case the GL_m -submodule generated by v is isomorphic to V_λ . The irreducible GL_m -module V_λ contains a unique (up to non-zero scalar multiples) vector fixed by U_m , and (up to non-zero scalar multiples), it is the only vector in V_λ on which \mathbb{T} acts by the weight λ .

The action of \mathbb{T} defines a \mathbb{Z}^m -grading on any GL_m -module M : $v \in M$ is *multihomogeneous of multidegree $\alpha = (\alpha_1, \dots, \alpha_m)$* if $\mathrm{diag}(z_1, \dots, z_m) \cdot v = (z_1^{\alpha_1} \dots z_m^{\alpha_m})v$ for all $\mathrm{diag}(z_1, \dots, z_m) \in \mathbb{T}$. In particular, $\mathbb{C}[V^m]$, $R_{n,m}$, $\mathcal{F}_{n,m}$ become \mathbb{Z}^m -graded algebras this way, and the map $\varphi : \mathcal{F}_{n,m} \rightarrow R_{n,m}$ is multihomogeneous. The polarized power sum $[x_1^{\alpha_1} \dots x_m^{\alpha_m}]$ is multihomogeneous of multidegree $(\alpha_1, \dots, \alpha_m)$.

The above \mathbb{Z}^m -grading is a refinement of the usual \mathbb{Z} -grading on the polynomial algebra $\mathbb{C}[V^m]$. Similarly, $R_{n,m}$ and $\mathcal{F}_{n,m}$ are graded algebras, the degree of a multihomogeneous element of multidegree $(\alpha_1, \dots, \alpha_m)$ being $\sum_{j=1}^m \alpha_j$. Denote by $\mathcal{F}^{(+)}$ the sum of homogeneous components of positive degree in the graded algebra $\mathcal{F} = \mathcal{F}_{n,m}^0$. Then $\mathcal{F}^{(+)}$ is a maximal ideal and $\mathcal{F}/\mathcal{F}^{(+)} \cong \mathbb{C}$. The ideal $\ker(\varphi) = \ker(\varphi_{n,m}^0)$ is homogeneous. By a *minimal system of generators of $\ker(\varphi)$* we mean a set of homogeneous elements that constitutes an irredundant generating system of the ideal $\ker(\varphi)$. It is well known that a subset $N \subset \ker(\varphi)$ of homogeneous elements is a minimal generating system of the ideal $\ker(\varphi)$ if and only if N is a basis of a \mathbb{C} -vector space direct complement of $\mathcal{F}^{(+)} \ker(\varphi)$ in $\ker(\varphi)$. This shows that although the minimal generating system N is not unique, for each degree the number of elements in N of that degree is uniquely determined. Even more, we may assume that N spans a GL_m -submodule $\mathrm{Span}\{N\}$ of \mathcal{F} , and the GL_m -module structure of $\mathrm{Span}\{N\}$ is uniquely determined by $\ker(\varphi)$. Indeed, note that an irreducible GL_m -submodule of \mathcal{F} isomorphic to V_λ is contained in the homogeneous component of \mathcal{F} of degree $|\lambda| = \sum_{j=1}^m \lambda_j$. Therefore we may take a GL_m -module direct complement of

$\mathcal{F}^{(+)} \ker(\varphi)$ in $\ker(\varphi)$, and a homogeneous \mathbb{C} -basis of this complement is a minimal generating system of $\ker(\varphi)$ with the desired properties.

Definition 2.3 For $\lambda \in \text{Par}_m$ denote by $\text{mult}_{n,m}(\lambda)$ the multiplicity of the irreducible GL_m -module V_λ as a summand in the factor GL_m -module $\ker(\varphi_{n,m}^n)/(\mathcal{F}_{n,m}^n)^{(+)} \cdot \ker(\varphi_{n,m}^n)$, and for a partition λ with more than m non-zero parts set $\text{mult}_{n,m}(\lambda) = 0$.

Next we recall that the multiplicities $\text{mult}_{n,m}(\lambda)$ have only a mild dependence on the parameter m . Note that a highest weight vector f of weight λ in $\mathcal{F}_{n,m}$ is in particular multihomogeneous of multidegree λ , hence is contained in the subalgebra $\mathcal{F}_{n,h(\lambda)}$.

Lemma 2.4 Let f be a multihomogeneous element of multidegree λ in $\mathcal{F}_{n,h(\lambda)}$, and let $m_1 \geq m_2 \geq h(\lambda)$ be positive integers. Then f is a highest weight vector for the action of GL_{m_1} on \mathcal{F}_{n,m_1} if and only if it is a highest weight vector for the action of GL_{m_2} on \mathcal{F}_{n,m_2} .

Proof. This is well known, and follows directly from the rule giving the action of the unipotent subgroup U_{m_1} (resp. U_{m_2}) on \mathcal{F}_{n,m_1} (resp. \mathcal{F}_{n,m_2}). \square

Corollary 2.5 Let λ be a partition. Then we have

$$\text{mult}_{n,m}(\lambda) = \begin{cases} \text{mult}_{n,h(\lambda)}(\lambda) & \text{if } m \geq h(\lambda); \\ 0 & \text{if } m < h(\lambda). \end{cases}$$

Proof. The case $m < h(\lambda)$ is trivial by definition of $\text{mult}_{n,m}(\lambda)$. Suppose $m > h(\lambda)$. Denote by $K_{n,m}$ the kernel of $\varphi : \mathcal{F}_{n,m}^n \rightarrow R_{n,m}$, and write $K_{n,h(\lambda)} := K_{n,m} \cap \mathcal{F}_{n,h(\lambda)}^n$ for the kernel of the restriction of φ to $\mathcal{F}_{n,h(\lambda)}^n$. Let N be a $\text{GL}_{h(\lambda)}$ -module complement of $(\mathcal{F}_{n,h(\lambda)}^n)^{(+)} K_{n,h(\lambda)}$ in $K_{n,h(\lambda)}$. Decompose N as a direct sum $\bigoplus N_i$ of irreducible $\text{GL}_{h(\lambda)}$ -modules, and take a highest weight vector f_i in N_i for all i . Then by Lemma 2.4, the f_i are highest weight vectors for the action of GL_m on $\mathcal{F}_{n,m}^n$. Moreover, taking into account the \mathbb{Z}^m -grading, one can easily show that $N \cap (\mathcal{F}_{n,m}^n)^{(+)} K_{n,m} = 0$, hence the f_i are linearly independent modulo $(\mathcal{F}_{n,m}^n)^{(+)} K_{n,m}$. This shows the inequality $\text{mult}_{n,h(\lambda)}(\lambda) \leq \text{mult}_{n,m}(\lambda)$. The proof of the reverse inequality is similar. \square

To simplify notation, write

$$\text{mult}_n(\lambda) := \text{mult}_{n,h(\lambda)}(\lambda).$$

The numbers $\text{mult}_n(\lambda)$ (all but finitely many of them are zero) carry all the sensible numerical information on a minimal generating system of the kernel of $\varphi : \mathcal{F}_{n,m}^n \rightarrow R_{n,m}$ by Corollary 2.5. Moreover, setting

$$h(\ker(\varphi)) := \max\{h(\lambda) \mid \text{mult}_n(\lambda) \neq 0\}$$

we have the following:

Corollary 2.6 The kernel of $\varphi : \mathcal{F}_{n,m}^n \rightarrow R_{n,m}$ is generated as a GL_m -stable ideal by its intersection with $\mathcal{F}_{n,h(\ker(\varphi))}^n$.

Proof. Let $\bigoplus_i M_i$ be a GL_m -module direct complement of $\mathcal{F}^{(+)} \ker(\varphi)$ in $\ker(\varphi)$, where the M_i are irreducible GL_m -modules. Let f_i be a highest weight vector in M_i . Then all the f_i belong to $\mathcal{F}_{n,h(\ker(\varphi))}^n$, and the GL_m -module $\bigoplus_i M_i$ generated by the set $\{f_i\}$ generates $\ker(\varphi)$ as an ideal. \square

3 Main results

First specialize to $n = 3$. The fundamental elements $\Psi(w_1, w_2, w_3, w_4) := \Psi_4(w_1, w_2, w_3, w_4)$ of $\ker(\varphi_{3,m})$ defined in (2) (here $w_1, w_2, w_3, w_4 \in \mathcal{M}_m$) take the form

$$\begin{aligned} \Psi(w_1, w_2, w_3, w_4) = & -6t(w_1w_2w_3w_4) \\ & + 2t(w_1w_2w_3)t(w_4) + 2t(w_1w_2w_4)t(w_3) + 2t(w_1w_3w_4)t(w_2) + 2t(w_2w_3w_4)t(w_1) \\ & + t(w_1w_2)t(w_3w_4) + t(w_1w_3)t(w_2w_4) + t(w_1w_4)t(w_2w_3) \\ & - t(w_1w_2)t(w_3)t(w_4) - t(w_1w_3)t(w_2)t(w_4) - t(w_1w_4)t(w_2)t(w_3) \\ & - t(w_2w_3)t(w_1)t(w_4) - t(w_2w_4)t(w_1)t(w_3) - t(w_3w_4)t(w_1)t(w_2) \\ & + t(w_1)t(w_2)t(w_3)t(w_4). \end{aligned}$$

Next we define an element $J_{3,2}$ and an element $J_{4,2}$ in $\mathcal{F}_{3,2}$; they have multidegree $(3, 2)$, and $(4, 2)$, respectively. To simplify notation, we write x, y instead of x_1, x_2 , so \mathcal{M}_2 consists of monomials in the commuting indeterminates x, y .

$$J_{3,2} := \frac{1}{2}(3\Psi(xy, x, x, y) - 3\Psi(x, x, x, y^2) + \Psi(x, x, x, y)t(y) - \Psi(x, x, y, y)t(x))$$

$$\begin{aligned} J_{4,2} := & 3\Psi(xy, xy, x, x) - 3\Psi(x, x, x, xy^2) + 2\Psi(x, x, x, y)t(xy) \\ & - \Psi(x, x, y, y)t(x^2) - \Psi(x, x, x, y^2)t(x) \end{aligned}$$

The elements $J_{3,2}$ and $J_{4,2}$ are both contained in $\mathcal{F}_{3,2}^3$ (i.e. they do not involve variables $t(w)$ with $\deg(w) > 3$). Indeed, direct calculation shows that

$$\begin{aligned} J_{3,2} = & 6t(x^2y)t(xy) - 3t(xy^2)t(x^2) - 2t(x^2y)t(x)t(y) \\ & + t(xy^2)t(x)^2 - 4t(xy)^2t(x) + 2t(xy)t(x)^2t(y) - 3t(x^3)t(y^2) \\ & + 4t(x^2)t(x)t(y^2) - t(x)^3t(y^2) + t(x^3)t(y)^2 - t(x^2)t(x)t(y)^2 \end{aligned}$$

and

$$\begin{aligned} J_{4,2} = & 6t(x^2y)^2 + t(xy)^2t(x^2) - 3t(xy)^2t(x)^2 - 6t(x^3)t(xy^2) \\ & + 2t(x^2)t(xy^2)t(x) + 4t(x^3)t(xy)t(y) \\ & - 2t(x^2)t(xy)t(x)t(y) + 2t(xy)t(x)^3t(y) - 4t(x^2y)t(x^2)t(y) \\ & - t(x^2)^2t(y^2) + t(x^2)^2t(y)^2 + 4t(x^2)t(x)^2t(y^2) \\ & - t(x^2)t(x)^2t(y)^2 - t(x)^4t(y^2) - 2t(x^3)t(x)t(y^2) \end{aligned}$$

Finally, denote by $J_{2,2,2}$ the element for $n = 3$ defined by (3) in general. Clearly, $J_{2,2,2}$ belongs to $\mathcal{F}_{3,3}^2$ and has multidegree $(2, 2, 2)$.

Theorem 3.1 *We have*

$$\text{mult}_3(\lambda) = \begin{cases} 1 & \text{for } \lambda = (3, 2), \quad \lambda = (4, 2), \quad \text{and} \quad \lambda = (2, 2, 2); \\ 0 & \text{for all other } \lambda. \end{cases}$$

For $m \geq 2$ the elements $J_{3,2}, J_{4,2} \in \mathcal{F}_{3,m}^3$ generate irreducible GL_m -submodules $N_{(3,2)}^m \cong V_{(3,2)}$, $N_{(4,2)}^m \cong V_{(4,2)}$ in $\ker(\varphi)$, and for $m \geq 3$ the element $J_{2,2,2} \in \mathcal{F}_{3,m}^3$ generates an irreducible GL_m -submodule $N_{(2,2,2)}^m \cong V_{(2,2,2)}$ in $\ker(\varphi)$. Furthermore, choose arbitrary \mathbb{C} -bases $\mathcal{G}_{(3,2)}^m, \mathcal{G}_{(4,2)}^m, \mathcal{G}_{(2,2,2)}^m$ in $N_{(3,2)}^m, N_{(4,2)}^m, N_{(2,2,2)}^m$. Set $\mathcal{G}^m := \mathcal{G}_{(3,2)}^m \cup \mathcal{G}_{(4,2)}^m \cup \mathcal{G}_{(2,2,2)}^m$ when $m \geq 3$ and $\mathcal{G}^2 := \mathcal{G}_{(3,2)}^2 \cup \mathcal{G}_{(4,2)}^2$. Then \mathcal{G}^m is a minimal generating system of the kernel of the surjection $\varphi : \mathcal{F}_{3,m}^3 \rightarrow R_{3,m}$ for any $m \geq 2$.

In classical language (see for example [16]), the elements in the GL_m -module generated by $f \in \ker(\varphi)$ are called the *polarizations of f* . So Theorem 3.1 can be paraphrased as follows: the polarizations of $J_{3,2}$, $J_{4,2}$, $J_{2,2,2}$ minimally generate the ideal of relations among the polarized power sums of degree at most three in dimension three.

For an arbitrary n we show that the ideal of relations between the polarized power sums can not be generated in degree strictly less than $2n$:

Theorem 3.2 *Suppose $m \geq n$. The element $J \in \ker(\varphi_{n,m}^n)$ given in (3) does not belong to the ideal $(\mathcal{F}_{n,m}^n)^{(+)} \cdot \ker(\varphi_{n,m}^n)$. In particular, denoting by $2^n := (2, \dots, 2) \in \mathrm{Par}_n$ the partition with n non-zero parts, all equal to 2, we have $\mathrm{mult}_n(2^n) = 1$.*

Denote by $\beta(n, m)$ the largest degree of an element in a minimal generating system of the ideal $\ker(\varphi_{n,m}^n)$ of relations between the polarized power sums. We summarize our present knowledge of $\beta(n, m)$. The general upper bound

$$\beta(n, m) \leq n^2 - n + 2$$

is pointed out in [6]. By Theorem 3.2 we have the general lower bound

$$\beta(n, m) \geq 2n \text{ for } m \geq n.$$

We have $\beta(2, m) = 4$ for all $m \geq 2$ (see for example [6]), and by Theorem 3.1 we have

$$\beta(3, m) = 6 \text{ for all } m \geq 2.$$

Note that both for $n = 2$ and $n = 3$ the exact value of $\beta(n, m)$ agrees with the general lower bound $2n$ established here. Moreover, both for $n = 2$ and $n = 3$ we have the equality $h(\ker(\varphi_{n,m}^n)) = n$ when $m \geq n$.

4 Reduction to $m = 4$

Proposition 4.1 *If V_λ occurs as an irreducible GL_m -module summand in the degree d homogeneous component of $\mathcal{F}_{n,m}^n$, then $h(\lambda) \leq \frac{d+1}{2}$.*

Proof. Denote by $S^k(M)$ the k th symmetric tensor power of the GL_m -module M . The degree d homogeneous component of $\mathcal{F}_{n,m}^n$ is isomorphic as a GL_m -module to

$$\bigoplus_{d_1+2d_2+3d_3+\dots+nd_n=d} S^{d_1}(V_{(1)}) \otimes S^{d_2}(V_{(2)}) \otimes \dots \otimes S^{d_n}(V_{(n)}). \quad (4)$$

Note that $S^{d_1}(V_{(1)}) \cong V_{(d_1)}$, and for $i \geq 2$, $S^{d_i}(V_{(i)})$ is a GL_m -submodule of $V_{(i)} \otimes \dots \otimes V_{(i)}$ (d_i tensor factors), which involves only summands V_λ with $h(\lambda) \leq d_i$ by Pieri's rule (I.5.16 in [12]). One concludes by the Littlewood-Richardson rule (I.9.2 in [12]) that for the irreducible constituents V_λ of (4) we have $h(\lambda) \leq 1 + d_2 + d_3 + \dots + d_n \leq \frac{d+1}{2}$ (the latter inequality follows from $d = \sum_{i=1}^n i d_i$). \square

Proposition 4.2 *We have the inequality $h(\ker(\varphi)) \leq (n^2 - n + 2)/2$. Consequently, the kernel of $\varphi : \mathcal{F}_{n,m}^n \rightarrow R_{n,m}$ is generated as a GL_m -stable ideal by its intersection with $\mathcal{F}_{n,(n^2-n+2)/2}^n$.*

Proof. We know from Proposition 2.1 that $\ker(\varphi)$ is generated as an ideal by the sum M of its homogeneous components of degree $\leq n^2 - n + 2$. Decompose M as the direct sum $\bigoplus_i M_i$ of irreducible GL_m -modules. By Proposition 4.1, $M_i \cong V_{\lambda_i}$ for some $\lambda_i \in \mathrm{Par}_m$ with $h(\lambda_i) \leq (n^2 - n + 2)/2$. Since M contains a GL_m -module direct complement of $(\mathcal{F}_{n,m}^n)^{(+)} \ker(\varphi)$ in the GL_m -module $\ker(\varphi)$, it follows that $\mathrm{mult}_{n,m}(\lambda) = 0$ when $h(\lambda) > \frac{n^2 - n + 2}{2}$. This shows the inequality $h(\ker(\varphi)) \leq n^2 - n + 2$, implying by Corollary 2.6 the second statement. \square

For $n = 3$ we have $\frac{n^2 - n + 2}{2} = 4$, hence by Proposition 4.2 it is sufficient to prove Theorem 3.1 in the special case $m = 4$.

5 Minimality

Throughout this section we assume $n = 3$. First we determine the GL_m -module structure of the kernel $K_{3,m}$ of $\varphi : \mathcal{F}_{3,m}^3 \rightarrow R_{3,m}$ up to degree 6. Denote by $K_{3,m}^{(d)}$ the degree d homogeneous component of $K_{3,m}$. Note that similarly to Corollary 2.5 one has that the multiplicity of V_λ as a summand in the $\mathrm{GL}_{h(\lambda)}$ -module $K_{3,h(\lambda)}$ is the same as the multiplicity of V_λ in the GL_m -module $K_{3,m}$ for an arbitrary $m \geq h(\lambda)$.

Proposition 5.1 *We have $K_{3,m}^{(d)} = 0$ for $d \leq 4$, and for $d = 5, 6$ the following GL_m -module isomorphisms hold:*

$$\begin{aligned} K_{3,m}^{(5)} &\cong V_{(3,2)} \quad \text{for } m \geq 2; \\ K_{3,m}^{(6)} &\cong 2 \cdot V_{(4,2)} + V_{(3,3)} + V_{(3,2,1)} + V_{(2,2,2)} \quad \text{for } m \geq 3. \end{aligned}$$

Proof. The fact that $K_{3,m}^{(d)} = 0$ for $d \leq 4$ follows for example from Proposition 2.1. Denote by $(\mathcal{F}_{3,m}^3)^{(d)}$ and $R_{3,m}^{(d)}$ the degree d homogeneous component of $\mathcal{F}_{3,m}^3$ and $R_{3,m}$. By formula (4) we have

$$\begin{aligned} (\mathcal{F}_{3,m}^3)^{(5)} &\cong S^5(V_{(1)}) + S^3(V_{(1)}) \otimes V_{(2)} + S^2(V_{(1)}) \otimes V_{(3)} + V_{(2)} \otimes V_{(3)} + V_{(1)} \otimes S^2(V_{(2)}) \\ &\cong V_{(5)} + 3 \cdot V_{(3)} \otimes V_{(2)} + V_{(1)} \otimes S^2(V_{(2)}), \end{aligned}$$

whereas

$$\begin{aligned} (\mathcal{F}_{3,m}^3)^{(6)} &\cong V_{(6)} + V_{(4)} \otimes V_{(2)} + V_{(3)} \otimes V_{(3)} + V_{(2)} \otimes S^2(V_{(2)}) \\ &\quad + V_{(1)} \otimes V_{(2)} \otimes V_{(3)} + S^3(V_{(2)}) + S^2(V_{(3)}). \end{aligned}$$

By Pieri's rule and some known plethysm formulae (see Section I.8 in [12]) one derives

$$(\mathcal{F}_{3,m}^3)^{(5)} \cong 5 \cdot V_{(5)} + 4 \cdot V_{(4,1)} + 4 \cdot V_{(3,2)} + V_{(2,2,1)}$$

and

$$(\mathcal{F}_{3,m}^3)^{(6)} \cong 7 \cdot V_{(6)} + 5 \cdot V_{(5,1)} + 8 \cdot V_{(4,2)} + V_{(4,1,1)} + 2 \cdot V_{(3,3)} + 2 \cdot V_{(3,2,1)} + 2 \cdot V_{(2,2,2)}.$$

To determine the GL_m -module structure of $R_{3,m}$ we start from the action of $\mathrm{GL}_3 \times \mathrm{GL}_m$ on $\mathbb{C}[V^m] = \mathbb{C}[x_{ij} \mid i = 1, 2, 3, \quad j = 1, \dots, m]$ by \mathbb{C} -algebra automorphisms given on the generators as follows: $(g, h) \in \mathrm{GL}_3 \times \mathrm{GL}_m$ sends x_{ij} to the (i, j) -entry of the matrix $g^T(x_{ij})_{i=1,2,3}^{j=1,\dots,m} h$. Cauchy's formula (I.4.3 in [12]) tells us the $\mathrm{GL}_3 \times \mathrm{GL}_m$ -module structure of $\mathbb{C}[V^m]$:

$$\mathbb{C}[V^m] \cong \bigoplus_{\lambda \in \mathrm{Par}_{\min\{3,m\}}} V_\lambda \otimes V_\lambda$$

Consequently, the multiplicity of V_λ in $R_{3,m}$ equals $\dim_{\mathbb{C}}(V_\lambda^{S_3})$, where we identify S_3 with the subgroup of permutation matrices in GL_3 , and we write $V_\lambda^{S_3}$ for the subspace of S_3 -fixed points in the GL_3 -module V_λ . By the Jacobi-Trudi Formula (I.3.4 in [12]), the character of an element $g \in S_3$ on V_λ (where $h(\lambda) \leq 3$) equals the determinant of the 3×3 matrix whose (i, j) -entry is $h_{\lambda_i - i + j}(z_1, z_2, z_3)$, where $h_k(z_1, z_2, z_3)$ is the k th complete symmetric polynomial in the eigenvalues z_1, z_2, z_3 of $g \in S_3 < \mathrm{GL}_3$. On the other hand, if $g \in S_3 < \mathrm{GL}_3$ has eigenvalues z_1, z_2, z_3 , then $h_k(z_1, z_2, z_3)$ equals the number of monomials of degree k in the variables x_1, x_2, x_3 fixed by g (where S_3 acts by permuting the variables). Based on this one can quickly compute the GL_m -module structure of $R_{3,m}$, and gets

$$R_{3,m}^{(5)} \cong 5 \cdot V_{(5)} + 4 \cdot V_{(4,1)} + 3 \cdot V_{(3,2)} + V_{(2,2,1)}$$

and

$$R_{3,m}^{(6)} \cong 7 \cdot V_{(6)} + 5 \cdot V_{(5,1)} + 6 \cdot V_{(4,2)} + V_{(4,1,1)} + V_{(3,3)} + V_{(3,2,1)} + V_{(2,2,2)}.$$

Since the multiplicity of V_λ in $K_{3,m}$ equals the difference of the multiplicities of V_λ in $\mathcal{F}_{3,m}^3$ and in $R_{3,m}$, the statement follows. \square

Proposition 5.2 *$J_{3,2}, J_{4,2}, J_{2,2,2}$ are highest weight vectors for the action of GL_m on $K_{3,m}$, and none of them is contained in the ideal $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$.*

Proof. Recall that the multidegree of any element of V_λ is lexicographically smaller than λ . Therefore Proposition 5.1 shows that the multihomogeneous component of multidegree $(3, 2)$ in $K_{3,m}$ is one-dimensional, and its non-zero elements are the highest weight vectors of the summand $V_{(3,2)}$. On the other hand, $J_{3,2}$ belongs to $K_{3,m}$ and has multidegree $(3, 2)$, so it is a highest weight vector. Moreover, since $K_{3,m}$ does not contain elements of degree less than five, we conclude that $J_{3,2}$ is not contained in $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$.

Similarly, an inspection of the decomposition of $K_{3,m}^{(6)}$ given in Proposition 5.1 shows that the multihomogeneous component of multidegree $(4, 2)$ is two-dimensional, and all its non-zero elements are highest weight vectors generating a submodule isomorphic to $V_{(4,2)}$. Consequently, $J_{4,2}$ is a highest weight vector.

By Proposition 2.2 we know that $g \cdot J_{2,2,2} = \det^2(g) J_{2,2,2}$ for any $g \in \mathrm{GL}_3$, hence in the special case $m = 3$, $J_{2,2,2}$ spans a one-dimensional GL_3 -submodule isomorphic to $V_{(2,2,2)}$. Consequently, $J_{2,2,2}$ is a highest weight vector for any $m \geq 3$ by Lemma 2.4.

Since the minimal degree of an element of $K_{3,m}$ is 5, we have

$$K_{3,m}^{(6)} \cap (\mathcal{F}_{3,m}^3)^{(+)}K_{3,m} = \sum_{j=1}^m t(x_j) K_{3,m}^{(5)}.$$

Note that $J_{4,2}$ contains the term $6t(x_1^2 x_2)^2$ and $J_{2,2,2}$ contains the term $3t(x_1 x_2)t(x_1 x_3)t(x_2 x_3)$. We conclude that none of them is contained in $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$. \square

Denote by $N_{(3,2)}^m, N_{(4,2)}^m, N_{(2,2,2)}^m$ the GL_m -submodules in $\mathcal{F}_{3,m}^3$ generated by $J_{3,2}, J_{4,2}, J_{2,2,2}$.

Corollary 5.3 *$N_{(3,2)}^m, N_{(4,2)}^m$, and $N_{(2,2,2)}^m$ are irreducible GL_m -submodules of $K_{3,m}$ isomorphic to $V_{(3,2)}, V_{(4,2)}$, and $V_{(2,2,2)}$. Moreover, the intersection of $N_{(3,2)}^m + N_{(4,2)}^m + N_{(2,2,2)}^m$ and the ideal $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$ is zero.*

Proof. Taking into account the multidegrees of $J_{3,2}, J_{4,2}, J_{2,2,2}$, the first statement immediately follows from Proposition 5.2. Moreover, $N_{(3,2)}^m, N_{(4,2)}^m, N_{(2,2,2)}^m$ are pairwise non-isomorphic irreducible GL_m -modules and none of them is contained in the GL_m -module $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$ (again by Proposition 5.2), hence their sum $N_{(3,2)}^m + N_{(4,2)}^m + N_{(2,2,2)}^m$ is disjoint from $(\mathcal{F}_{3,m}^3)^{(+)}K_{3,m}$ by basic principles of representation theory. \square

Choose arbitrary \mathbb{C} -bases $\mathcal{G}_{(3,2)}^m, \mathcal{G}_{(4,2)}^m$, and $\mathcal{G}_{(2,2,2)}^m$ in $N_{(3,2)}^m, N_{(4,2)}^m$, and $N_{(2,2,2)}^m$, and set $\mathcal{G}^m := \mathcal{G}_{(3,2)}^m \cup \mathcal{G}_{(4,2)}^m \cup \mathcal{G}_{(2,2,2)}^m$. Then by Corollary 5.3, \mathcal{G}^m can be extended to a minimal system of generators of the ideal $K_{3,m}$. To prove that \mathcal{G}^m is actually a minimal system of generators of $K_{3,m}$, it is sufficient to show that the ideal $K_{3,m}$ can be generated by $|\mathcal{G}^m|$ elements.

Recall that the dimension of the GL_m -module V_λ (where $\lambda \in \mathrm{Par}_m$) equals

$$d_\lambda(m) := \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

by the Weyl dimension formula (see for example (7.1.17) in [7]), and so

$$|\mathcal{G}^m| = d_{(3,2)}(m) + d_{(4,2)}(m) + d_{(2,2,2)}(m).$$

6 Hironaka decomposition

It is well known that

$$P := \{[x_j], [x_j^2], [x_j^3] \mid j = 1, \dots, m\}$$

is a *homogeneous system of parameters* in $R_{3,m}$. Write $\mathbb{C}[P]$ for the \mathbb{C} -subalgebra of R generated by P . It is a polynomial ring in the $3m$ generators, and R is a finitely generated $\mathbb{C}[P]$ -module. Moreover, since R is Cohen-Macaulay, it is a free $\mathbb{C}[P]$ -module. A set $\mathbf{S} \subset R_{3,m}$ of homogeneous elements constitutes a free $\mathbb{C}[P]$ -module generating system of R if and only if the image of \mathbf{S} is a \mathbb{C} -vector space basis of the factor algebra $R/(P)$ of R modulo the ideal (P) generated by P . The elements of P (respectively \mathbf{S}) are referred to as the *primary* (respectively *secondary*) generators of $R_{3,m}$, and

$$R_{3,m} = \bigoplus_{s \in \mathbf{S}} \mathbb{C}[P] \cdot s \quad (5)$$

the *Hironaka decomposition* of $R_{3,m}$. Write $Q := \{[x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \mid \alpha_1 + \cdots + \alpha_m \leq 3\}$ for the chosen minimal \mathbb{C} -algebra generating system of $R_{3,m}$. We have $Q \supset P$, and we may assume that \mathbf{S} consists of products of powers of the elements of Q (in particular, then \mathbf{S} consists of multihomogeneous elements, and the empty product $1 \in \mathbf{S}$). Recall that the Hilbert series of an \mathbb{N}_0^m -graded vector space $A := \bigoplus_\alpha A^\alpha$ with $\dim_{\mathbb{C}}(A^\alpha) < \infty$ is the formal power series in $\mathbb{Z}[[t_1, \dots, t_m]]$ defined by

$$H(A; t_1, \dots, t_m) := \sum_{\alpha = (\alpha_1, \dots, \alpha_m)} \dim_{\mathbb{C}}(A^\alpha) t_1^{\alpha_1} \cdots t_m^{\alpha_m}.$$

It follows from (5) that

$$H(R_{3,m}; t_1, \dots, t_m) = \frac{H(\mathrm{Span}_{\mathbb{C}}(\mathbf{S}); t_1, \dots, t_m)}{\prod_{j=1}^m (1 - t_j)(1 - t_j^2)(1 - t_j^3)} \quad (6)$$

where $\mathrm{Span}_{\mathbb{C}}(\mathbf{S})$ is the \mathbb{C} -subspace in R spanned by \mathbf{S} (since \mathbf{S} consists of multihomogeneous elements, it is \mathbb{N}_0^m -graded). On the other hand, the Hilbert series of R can be explicitly calculated

(see Section 7), and from this we know the number of elements of \mathbf{S} having multidegree α for each α .

The following two statements provide our basis to find a complete system of relations.

Lemma 6.1 *Fix a positive integer d , and let S be a finite set of monomials in the elements of Q , each element of S having degree at most d , and suppose that S satisfies the following:*

- (i) $1 \in S$, and $Q \setminus P \subseteq S$;
- (ii) *For each $e = 1, \dots, d$, the number of degree e elements in S equals the number of degree e elements in a system of secondary generators of $R_{3,m}$.*
- (iii) *For any $s \in S$ and $q \in Q \setminus P$ with $\deg(s \cdot q) \leq d$ there exist scalars $\gamma_a \in \mathbb{C}$ ($a \in S$, $\deg(a) = \deg(sq)$) with*

$$s \cdot q - \sum_{\deg(a)=\deg(sq)} \gamma_a a \in (P). \quad (7)$$

Then S can be extended to a system \mathbf{S} of secondary generators of $R_{3,m}$ such that S coincides with the subset of degree $\leq d$ elements of \mathbf{S} .

Proof. A straightforward induction on the degree. □

We shall use the following notation: for $f_1, f_2 \in R$, we write $f_1 \equiv f_2$ if $f_1 - f_2 \in (P)$. For example, (7) reads as

$$s \cdot q \equiv \sum_{\deg(a)=\deg(sq)} \gamma_a a,$$

and it means that there exists a multihomogeneous element r_s^q in the kernel $K_{3,m}$ of the surjection $\varphi : \mathcal{F}_{3,m}^3 \rightarrow R_{3,m}$ such that

$$r_s^q - s^* \cdot q^* + \sum_{\deg(a)=\deg(sq)} \gamma_a a^* \in \sum_{j=1}^m \sum_{k=1}^3 \mathcal{F}_{3,m}^3 \cdot t(x_j^k) \quad (8)$$

where given a product $c = [w_1] \cdots [w_l]$ of the generators $[w_i] \in Q$ we write $c^* := t(w_1) \cdots t(w_l) \in \mathcal{F}$. (Of course, r_s^q is not unique, it is determined modulo the intersection of $K_{3,m}$ and the ideal on the right hand side of (8).)

Lemma 6.2 *Suppose that the assumptions of Lemma 6.1 hold. For all $s \in S$, $q \in Q \setminus P$ with $\deg(qs) \leq d$ choose an element r_s^q satisfying (8). Then the ideal generated by the r_s^q contains all homogeneous components of $K_{3,m}$ up to degree d .*

Proof. Straightforward. □

We shall use also the special case $n = 3$ of Lemma 6.1 from [6]:

Lemma 6.3 *If the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \mathcal{M}_m$ has degree at least 3 in one of the variables x_1, \dots, x_m , and $\alpha_1 + \cdots + \alpha_m \geq 4$, then $[x_1^{\alpha_1} \cdots x_m^{\alpha_m}] \equiv 0$.*

7 Hilbert series

In this section we express the Hilbert series of $R_{3,m}$ in a form that is practical to evaluate for small m . The symmetric group S_3 has three irreducible complex characters: χ_0 , the trivial character, χ_1 , the character of the 2-dimensional irreducible representation, and χ_2 , the sign character. Denote by C the *character ring* of S_3 ; i.e. C is the subring of the algebra of central functions on S_3 generated by χ_0, χ_1, χ_2 . It is a free \mathbb{Z} -module spanned by χ_0, χ_1, χ_2 . The multiplication in C is given as follows: χ_0 is the identity element, $\chi_1^2 = \chi_0 + \chi_1 + \chi_2$, $\chi_2^2 = \chi_0$, $\chi_1 \cdot \chi_2 = \chi_1$. For a graded S_3 -module $A := \bigoplus_{k=0}^{\infty} A^{(k)}$ we set $H_{\chi_i}(A; t) := \sum_{k=0}^{\infty} \text{mult}_{\chi_i}(A^{(k)}) t^k$, where $\text{mult}_{\chi_i}(A^{(k)})$ denotes the multiplicity of the irreducible representation with character χ_i as a summand of $A^{(k)}$. Moreover, set

$$H_{S_3}(A; t) := \sum_{i=0}^2 \chi_i H_{\chi_i}(A; t) \in C[[t]].$$

One defines the Hilbert series of a multigraded S_3 -module in a similar way. Clearly, the Hilbert series of the multigraded vector space R coincides with the coefficient of χ_0 in

$$H_{S_3}(\mathbb{C}[V^m]; t_1, \dots, t_m) \in \sum_{i=0}^2 \mathbb{Z}[[t_1, \dots, t_m]] \chi_i.$$

We have the isomorphism $\mathbb{C}[V^m] \cong \mathbb{C}[V] \otimes \dots \otimes \mathbb{C}[V]$, hence

$$H_{S_3}(\mathbb{C}[V^m]; t_1, \dots, t_m) = \prod_{j=1}^m H_{S_3}(\mathbb{C}[V]; t_j)$$

where multiplication is understood in the ring of formal power series $C[[t_1, \dots, t_m]]$ with coefficients in the character ring C of S_3 . It is well known that

$$H_{S_3}(V; t) = \frac{\chi_0 + (t + t^2)\chi_1 + t^3\chi_2}{(1-t)(1-t^2)(1-t^3)}.$$

Taking into account (6) we conclude that the Hilbert series of a system of multihomogeneous secondary generators \mathbf{S} (defined in Section 6) equals the coefficient of χ_0 in

$$\prod_{j=1}^m (\chi_0 + (t_j + t_j^2)\chi_1 + t_j^3\chi_2) \in \sum_{i=0}^2 \mathbb{Z}[[t_1, \dots, t_m]] \chi_i. \quad (9)$$

8 The cases $m = 2, 3, 4$

In this section we prove that the kernel $K_{3,4}$ of the surjection $\varphi : \mathcal{F}_{3,4}^3 \rightarrow R_{3,4}$ can be generated by

$$d_{(3,2)}(4) + d_{(4,2)}(4) + d_{(2,2,2)}(4) = 60 + 126 + 10 = 196$$

elements. By the concluding remarks in Section 5, it follows that Theorem 3.1 holds in the special case $m = 4$. This finishes the proof of Theorem 3.1 for arbitrary m by the concluding remark of Section 4.

To simplify notation, we shall write x, y, z, w instead of x_1, x_2, x_3, x_4 .

8.1 The case $m = 2$

(This case is sketched in [6].) By (9) we have

$$H(\mathbf{S}; t, u) = 1 + tu + t^2u + tu^2 + t^2u^2 + t^3u^3.$$

Set

$$S := \{1, [xy], [x^2y], [xy^2], [xy]^2, [x^2y][xy^2]\}.$$

The equality $\varphi(\Psi(x, x, y, y)) = 0$ yields the congruence

$$[x^2y^2] \equiv \frac{1}{3}[xy]^2 \quad (10)$$

and the equality $\varphi(\Psi(xy, x, x, y)) = 0$ yields

$$6[xy \cdot x \cdot x \cdot y] \equiv 4[x^2y][xy].$$

It follows by Lemma 6.3 that

$$[x^2y][xy] \equiv 0. \quad (11)$$

As explained before Lemma 6.2, to the congruence (11) there belongs an element $r_{[x^2y]}^{[xy]} \in K_{3,2}$. By symmetry in x and y , we have also the congruence and the corresponding relation:

$$[xy^2][xy] \equiv 0 \quad \text{implied by} \quad r_{[xy^2]}^{[xy]} \in K_{3,2}.$$

We have the congruence

$$6[xy \cdot xy \cdot x \cdot x] \equiv 4[x^3y][xy] + 2[x^2y]^2$$

(obtained by substituting the factors of $xy \cdot xy \cdot x \cdot x$ on the left hand side into Ψ), yielding by Lemma 6.3

$$[x^2y]^2 \equiv 0 \quad \text{and} \quad r_{[x^2y]}^{[x^2y]} \in K_{3,2}.$$

By symmetry, $[xy^2]^2 \equiv 0$ from $r_{[xy^2]}^{[xy^2]} \in K_{3,2}$. Finally, we have

$$6[xy \cdot xy \cdot x \cdot y] \equiv 5[x^2y^2][xy] + 2[x^2y][xy^2] - [xy]^3$$

and taking into account Lemma 6.3 and (10) we get

$$[xy]^3 \equiv -3[x^2y][xy^2] \quad \text{and} \quad r_{[xy][xy]}^{[xy]}.$$

Clearly we can choose $r_{[xy]}^{[xy]} = 0$. Multiplying the congruence (11) by $[xy^2]$ we get $[xy]([x^2y][xy^2]) \equiv 0$, hence we can choose

$$r_{[x^2y][xy^2]}^{[xy]} := t(xy^2) \cdot r_{[x^2y]}^{[xy]}.$$

Similarly, it is easy to see that for the remaining $s \in S$ and $q \in Q \setminus P$ the element r_s^q can be chosen from the ideal generated by the 5 elements of $K_{3,2}$ introduced already. It follows by Lemma 6.1 that S is a system of secondary generators of $R_{3,2}$, and by Lemma 6.2 the ideal $K_{3,2}$ is generated by

$$r_{[x^2y]}^{[xy]}, \quad r_{[xy^2]}^{[xy]}, \quad r_{[x^2y]}^{[x^2y]}, \quad r_{[xy][xy]}^{[xy]}, \quad r_{[xy^2]}^{[xy^2]}.$$

Moreover, since $d_{(3,2)}(2) + d_{(4,2)}(2) = 2 + 3 = 5$, the above is a minimal system of generators of the ideal $K_{3,2}$.

8.2 The case $m = 3$

In Table 1 we collect the monomials in the elements of $Q \setminus P$ of descending multidegree α with all $\alpha_i > 0$, up to total degree 8. Monomials congruent to 0 are indicated by \star (and we indicate by \star the multidegrees where all monomials are congruent to 0), and the symbol \sim indicates that some non-zero scalar multiples of the given monomials are congruent modulo (P) . Table 2 should be interpreted as follows: its second line for example says that in multidegree $(3, 1, 1)$ we have the congruence $[x^2y][xz] + [x^2z][xy] \equiv 0$. Hence we may choose a multihomogeneous element $r_{3,1,1} \in K_{3,3}$ of multidegree $(3, 1, 1)$ differing from $t(x^2y)t(xz) + t(x^2z)t(xy)$ by an element of the ideal of $\mathcal{F}_{3,3}^3$ generated by $t(x^i), t(y^i), t(z^i)$, $i = 1, 2, 3$. (From now on we change the notation for the relations, the lower indices indicate their multidegree.)

Note that $\text{Span}_{\mathbb{C}}(P)$ and the ideal (P) are not GL_m -submodules in $R_{n,m}$. However, they are S_m -submodules, where we think of the symmetric group S_m as the subgroup of GL_m consisting of permutation matrices. Observe that some of the congruence in Table 2 are symmetric or skew symmetric in two variables, and some is symmetric in x, y, z . We may assume that the corresponding elements $r_{3,1,1}, r_{2,2,1}^{(1)}$, etc. are chosen so that they also have the corresponding symmetry or skew-symmetry. The last two columns of Table 2 contain the number of S_3 -translates (resp. S_4 -translates) of the relation listed in the third column, where we count the S_m -translates up to non-zero scalar multiples, so by the above observation the number of S_m -translates of r equals the index in S_m of the stabilizer of the congruence corresponding to r .

Denote by \mathcal{G} the relations listed in the third column of Table 2 and all their S_3 -translates. Note that the sum of the numbers in the last but one column of Table 2 equals the cardinality of \mathcal{G} , so $|\mathcal{G}| = 43$.

Using the S_3 -translates of the relations in Table 2 one can easily justify the \star symbols and the equivalences \sim in Table 1. This means that up to total degree 8, all monomials (having descending multidegree) in $Q \setminus P$ can be reduced to linear combinations of the monomials given in Table 3. (For multidegrees with $\alpha_3 = 0$, this was shown already in section 8.1.) One can easily see from (9) that for each descending multidegree α , the number of elements in Table 3 with multidegree α coincides with the coefficient of $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3}$ in $H(\mathbf{S}; t_1, t_2, t_3)$. Define S as follows: in descending multidegrees its elements are listed in Table 3, and if β is a multidegree in the S_3 -orbit of some descending multidegree α , then choose a permutation $\pi \in S_3$ with $\beta_i = \alpha_{\pi(i)}$, and include in S the images under π of the elements of multidegree α in Table 3. (Of course, the set S is not uniquely defined: for certain multidegrees, say for multidegree $(1, 3, 1)$ we may choose for π the transposition (12) or the three-cycle (123) . However, this does not influence the arguments below.) Since the set \mathcal{G} is (essentially) S_3 -stable, it follows that up to degree ≤ 8 , all monomials in $Q \setminus P$ can be reduced to linear combinations of S using the relations in \mathcal{G} . Moreover, $H(S; t_1, t_2, t_3) = H(\mathbf{S}; t_1, t_2, t_3)$. Consequently, by Lemmas 6.1 and 6.2, S is a system of secondary generators, and \mathcal{G} generates the ideal $K_{3,3}$ up to degree 8. We know from Proposition 2.1 that $K_{3,3}$ is generated in degree ≤ 8 , hence \mathcal{G} generates $K_{3,3}$. Since $d_{(3,2)}(3) + d_{(4,2)}(3) + d_{(2,2,2)}(3) = 15 + 27 + 1 = 43 = |\mathcal{G}|$, we conclude that \mathcal{G} is a minimal system of generators of the ideal $K_{3,3}$.

We finish this Section with the proof of the congruences in Table 2. The relations $r_{3,2}, r_{4,2}, r_{3,3}$ were explained in section 8.1. The relation $\varphi(\Psi(w_1, w_2, w_3, w_4)) = 0$ implies a congruence of multidegree multideg($w_1 w_2 w_3 w_4$) of the form

$$[w_1 \cdot w_2 \cdot w_3 \cdot w_4] \equiv \cdots$$

where \cdots is a linear combination of monomials in elements $[u]$ with $\deg(u) < \deg(w_1 w_2 w_3 w_4)$, which can be written as a polynomial in the elements of $Q \setminus P$ using (12) below.

$r_{3,1,1}$: $\varphi(\Psi(x^2, x, y, z)) = 0$ and $[x^3yz] \equiv 0$ (by Lemma 6.3) imply

$$0 \equiv 6[x^3yz] = 6[x^2 \cdot x \cdot y \cdot z] \equiv [x^2y][xz] + [x^2z][xy].$$

$r_{2,2,1}^{(1)}, r_{2,2,1}^{(2)}$: Eliminate $[x^2y^2z]$ from the following consequences of the fundamental relation:

$$6[x \cdot y \cdot z \cdot xy] \equiv 3[xy][xyz] + [xz][xy^2] + [x^2y][yz]$$

$$6[x^2 \cdot y \cdot y \cdot z] \equiv 2[x^2y][yz]$$

$$6[x \cdot x \cdot y^2 \cdot z] \equiv 2[xy^2][xz]$$

$r_{4,1,1}$: We have $0 \equiv [x^4yz] = [x^2 \cdot x^2 \cdot y \cdot z] \equiv \frac{1}{3}[x^2y][x^2z]$.

$r_{3,2,1}^{(2)}$: Follows by $0 \equiv 6[x^3y^2z] = 6[x^2 \cdot x \cdot y^2 \cdot z] \equiv [x^2y^2][xz] + [x^2z][xy^2]$ and (10).

$r_{3,2,1}^{(1)}$: The congruences $0 \equiv 6[x^3y^2z] = 6[xyz \cdot x \cdot x \cdot y] \equiv 2[x^2y][xyz] + 2[x^2yz][xy]$, and

$$[x^2yz] \equiv \frac{1}{3}[xy][xz] \quad (12)$$

yield $[xy]^2[xz] + 3[xyz][x^2y] \equiv 0$, and this and $r_{3,2,1}^{(2)}$ imply $r_{3,2,1}^{(1)}$.

$r_{2,2,2}^{(1)}, r_{2,2,2}^{(2)}$: Eliminate $[x^2y^2z^2]$ from the congruences

$$6[x \cdot x \cdot y^2 \cdot z^2] \equiv 2[xy^2][xz^2]$$

$$6[x^2 \cdot y \cdot y \cdot z^2] \equiv 2[x^2y][yz^2]$$

$$6[xyz \cdot x \cdot y \cdot z] \equiv 2[xyz]^2 + [x^2yz][yz] + [xy^2z][xz] + [xyz^2][xy]$$

$$6[xy \cdot x \cdot y \cdot z^2] \equiv 3[xy][xyz^2] + [x^2y][yz^2] + [xy^2][xz^2]$$

and use (12).

8.3 The case $m = 4$

The arguments are the same as in section 8.2. We just give the corresponding tables and prove the new congruences (involving all the four variables). Tables 4, 5 and 6 deal only with multidegrees α with all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, since the remaining multidegrees have been taken care in section 8.2.

The congruence in Table 4 corresponding to $r_{2,1,1,1}$ is symmetric in the variables z, w , hence the number of its S_4 -translates is $\frac{24}{2} = 12$. The same holds for $r_{3,1,1,1}$ and $r_{2,2,1,1}^{(3)}$. The congruence corresponding to $r_{2,2,1,1}^{(1)}$ is symmetric in the variables x, y and also in the variables z, w , hence the number of its S_4 -translates is $\frac{24}{2 \cdot 2} = 6$. The congruence corresponding to $r_{2,2,1,1}^{(2)}$ is unchanged if we simultaneously interchange x, y and z, w , hence the number of its S_4 -translates is $\frac{24}{2} = 12$.

The set \mathcal{G} of all S_4 -translates of the relations listed in Tables 2 and 4 has cardinality $|\mathcal{G}| = 196$. (This agrees with $d_{(3,2)}(4) + d_{(4,2)}(4) + d_{(2,2,2)} = 60 + 126 + 10 = 196$.) Table 5 together with Table 3 give a system of secondary generators up to degree 8 in descending multidegrees.

An inspection of Tables 4 and 6 shows that up to degree 8, all monomials in the elements of $Q \setminus P$ with descending multidegree can be reduced to a linear combination of elements listed

in Table 5 using the S_4 -translates of the relations in Tables 2 and 4. We just give some sample examples:

$$[x^2y][xz][xw] \equiv -[x^2z][xy][xw] \equiv [x^2w][xy][xz] \equiv -[x^2y][xz][xw]$$

by the relations $r_{3,1,1}$, $r_{3,0,1,1}$, $r_{3,1,0,1}$, hence all the above products are congruent to 0. The relations \sim in multidegree $(3, 3, 1, 1)$ can be derived as follows (at each congruence we indicate the relation whose S_m -translate is used):

$$\begin{aligned} & -\frac{1}{3}[xy]^2[xz][yw] \stackrel{r_{3,2,1}^{(1)}, r_{3,2,1}^{(2)}}{\equiv} [x^2y][xyz][yw] \stackrel{r_{3,2,1}^{(1)}}{\equiv} [x^2z][xy^2][yw] \stackrel{r_{3,1,1}}{\equiv} -[x^2z][y^2w][xy] \\ & \stackrel{r_{3,1,1}}{\equiv} [x^2y][y^2w][xz] \stackrel{r_{3,2,1}^{(1)}}{\equiv} [xy^2][xyw][xz] \stackrel{r_{2,2,1}^{(1)}}{\equiv} [x^2y][xyw][yz] \stackrel{r_{3,2,1}^{(1)}}{\equiv} [x^2w][xy^2][yz] \\ & \stackrel{r_{3,1,1}}{\equiv} -[x^2w][y^2z][xy] \stackrel{r_{3,1,1}}{\equiv} [x^2y][y^2z][xw] \stackrel{r_{3,2,1}^{(1)}}{\equiv} [xy^2][xyz][xw] \stackrel{r_{3,2,1}^{(1)}, r_{3,2,1}^{(2)}}{\equiv} -\frac{1}{3}[xy]^2[xw][yz] \end{aligned}$$

Furthermore,

$$-[xy]^3[zw] \stackrel{r_{3,3}}{\equiv} 3[x^2y][xy^2][zw] \stackrel{r_{2,1,1,1}}{\equiv} 3[xyz][xy^2][xw] + 3[xyw][xy^2][xz]$$

hence by the above long chain of congruences we conclude

$$[xy]^3[zw] \equiv -6[x^2y][xyz][yw].$$

Finally we verify the four-variable relations.

$r_{2,1,1,1}$: Various substitutions into the fundamental relation yield

$$6[x^2 \cdot y \cdot z \cdot w] \equiv [x^2y][zw] + [x^2z][yw] + [x^2w][yz] \quad (13)$$

$$6[x \cdot xy \cdot z \cdot w] \equiv [x^2y][zw] + 2[xy][xzw] + [xz][xyw] + [xw][xyz] \quad (14)$$

$$6[x \cdot y \cdot xz \cdot w] \equiv [x^2z][yw] + [xy][xzw] + 2[xz][xyw] + [xw][xyz] \quad (15)$$

$$6[x \cdot y \cdot z \cdot xw] \equiv [x^2w][yz] + [xy][xzw] + [xz][xyw] + 2[xw][xyz] \quad (16)$$

Now $\frac{1}{2}((13) + (14) - (15) - (16))$ gives $r_{2,1,1,1}$.

$r_{2,2,1,1}^{(1)}$, $r_{2,2,1,1}^{(2)}$, $r_{2,2,1,1}^{(3)}$: To make calculations more transparent, introduce the following temporary notation for the monomials of $Q \setminus P$ of multidegree $(2, 2, 1, 1)$: $a_1 = [xy]^2[zw]$, $a_2 = [xy][xz][yw]$, $a_3 = [xy][yz][xw]$, $b_1 = [x^2y][yzw]$, $b_2 = [xy^2][xzw]$, $c_1 = [x^2z][y^2w]$, $c_2 = [x^2w][y^2z]$, $d = [xyz][xyw]$. Using (10), (12), their S_4 -translates, and

$$6[xyzw] \equiv [xy][zw] + [xz][yw] + [xw][yz] \quad (17)$$

various substitutions into the fundamental relation yield

$$6[xy \cdot xz \cdot y \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_2 + d \quad (18)$$

$$6[xy \cdot x \cdot yz \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + b_1 + d \quad (19)$$

$$6[x^2y \cdot y \cdot z \cdot w] \equiv \frac{1}{3}a_1 + \frac{1}{3}a_2 + \frac{1}{3}a_3 + 2b_1 \quad (20)$$

$$6[xy \cdot xy \cdot z \cdot w] \equiv \frac{2}{3}a_2 + \frac{2}{3}a_3 + 2d \quad (21)$$

$$6[xz \cdot x \cdot yw \cdot y] \equiv \frac{1}{6}a_1 + \frac{1}{2}a_2 + \frac{1}{6}a_3 + c_1 + d \quad (22)$$

$$6[xw \cdot x \cdot yz \cdot y] \equiv \frac{1}{6}a_1 + \frac{1}{6}a_2 + \frac{1}{2}a_3 + c_2 + d \quad (23)$$

Equations (18), (19), (20) imply

$$b_1 \equiv b_2 \equiv d \quad (24)$$

(in particular, relation $r_{2,2,1,1}^{(3)}$). From equations (24), (18), and (21) we conclude

$$a_1 \equiv a_2 + a_3. \quad (25)$$

Taking the difference of (21) and (22), and eliminating a_1 by (25) we get

$$\frac{1}{3}a_3 \equiv c_1 - d. \quad (26)$$

Taking the difference of (21) and (23), and eliminating a_1 by (25) we get

$$\frac{1}{3}a_2 \equiv c_2 - d, \text{ hence } r_{2,2,1,1}^{(2)}. \quad (27)$$

Finally, (25), (26), (27) yield

$$a_1 \equiv 3c_1 + 3c_2 - 6d, \text{ hence } r_{2,2,1,1}^{(1)}.$$

$r_{3,1,1,1}^\bullet$:

$$0 \equiv 6[x^3yzw] = 6[x \cdot xy \cdot xz \cdot w] \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2z][xyw] \quad (28)$$

Permuting y, z, w we get

$$0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2y][xzw] + [x^2w][xyz] \quad (29)$$

$$0 \equiv \frac{2}{3}[xy][xz][xw] + [x^2z][xyw] + [x^2w][xyz] \quad (30)$$

Now (28) + (29) - (30) gives $r_{3,1,1,1}$.

9 The proof of Theorem 3.2

First we point out that the kernel $\ker(\varphi_{n,m}^2)$ of the restriction of φ to $\mathcal{F}_{n,m}^2$ is described by the *second fundamental theorem for vector invariants of the full orthogonal group* (cf. Theorem 2.17.A in [16]).

Proposition 9.1 *The ideal $\ker(\varphi_{n,m}^2)$ is (minimally) generated by the GL_m -submodule of $\mathcal{F}_{n,m}^2$ spanned by J .*

Proof. Denote by $\mathrm{O}(V)$ the orthogonal group, i.e. $\mathrm{O}(V)$ consists of the linear transformations of $V = \mathbb{C}^n$ preserving the standard quadratic form $(v_1, \dots, v_n) \mapsto \sum_{i=1}^n v_i^2$. The orthogonal complement V_0 of $(1, \dots, 1) \in V$ consists of the vectors in V with zero coordinate sum. We identify the stabilizer of $(1, \dots, 1)$ in $\mathrm{O}(V)$ with $\mathrm{O}(V_0)$ in the obvious way. Note that the elements of S_n as transformations on V do belong to $\mathrm{O}(V_0)$. As an immediate corollary of the *first fundamental theorem on vector invariants of the orthogonal group* (cf. Theorem 2.11.A in [16]) we conclude that $\varphi(\mathcal{F}_{n,m}^2) = \mathbb{C}[V^m]^{\mathrm{O}(V_0)} \subset R_{n,m}$. Set $b_{ij} := [x_i x_j] - \frac{1}{n}[x_i][x_j]$ for $1 \leq i, j \leq m$. The projection from $V \rightarrow V_0$ with kernel spanned by $(1, \dots, 1)$ induces an identification of $\mathbb{C}[V_0^m]^{\mathrm{O}(V_0)}$ with the subalgebra of $\mathbb{C}[V^m]^{\mathrm{O}(V_0)}$ generated by the b_{ij} . Moreover, $\mathbb{C}[V^m]^{\mathrm{O}(V_0)}$ is an m -variable polynomial ring over $\mathbb{C}[V_0^m]^{\mathrm{O}(V_0)}$ generated by $[x_1], \dots, [x_m]$. Denote by L the subalgebra of $\mathcal{F}_{n,m}^2$ generated by $u_{ij} := t(x_i x_j) - \frac{1}{n}t(x_i)t(x_j)$, $1 \leq i, j \leq n$. By the above considerations, the ideal $\ker(\varphi_{n,m}^2)$ is generated by the kernel of the restriction $\varphi|_L$ of φ to L . Now the kernel of $\varphi|_L : L \rightarrow \mathbb{C}[V_0^m]^{\mathrm{O}(V_0)}$, $u_{ij} \mapsto b_{ij}$ is given by Theorem 2.17.A in [16], stating that it is generated by the polarizations of J . \square

Let I be a highest weight vector in $\ker(\varphi)$ of weight $2^n = (2, \dots, 2)$. We claim that I necessarily belongs to $\mathcal{F}_{n,m}^2$, and if I is contained in $\mathcal{F}^{(+)} \cdot \ker(\varphi)$, then I is necessarily contained in $(\mathcal{F}_{n,m}^2)^{(+)} \cdot \ker(\varphi_{n,m}^2)$. It is clear that Theorem 3.2 follows from this claim and Proposition 9.1.

To prove this claim, given a polynomial GL_m -module U and a partition $\lambda \in \mathrm{Par}_m$, denote by $\lambda(U)$ the λ -isotypic component of U (i.e. the sum of the GL_m -submodules of U isomorphic to the irreducible GL_m -module V_λ). Write $\lambda \subset \mu$ (where $\lambda, \mu \in \mathrm{Par}_m$) if $\lambda_i \leq \mu_i$ for $i = 1, \dots, m$. It follows from Pieri's rule that denoting by A the ideal in \mathcal{F} generated by the $t(w)$ with $\deg(w) \geq 3$, we have $A \subset \sum_{(3) \subset \lambda} \lambda(\mathcal{F})$. Since $\mathcal{F}_{n,m}^2$ is a GL_m -module direct complement of A , the 2^n -isotypic component of \mathcal{F} is contained in $\mathcal{F}_{n,m}^2$. In particular, I belongs to $\mathcal{F}_{n,m}^2$.

Again by Pieri's rule, the ideal generated by $\lambda(\mathcal{F})$ is contained in $\sum_{\lambda \subset \mu} \mu(\mathcal{F})$. So if $I \in \mathcal{F}^{(+)} \cdot \ker(\varphi)$, then $I \in \sum_{\lambda \subsetneq 2^n} \mathcal{F}^{(+)} \lambda(\ker(\varphi))$. Again since $\mathcal{F}_{n,m}^2$ is a GL_m -module complement of A , we conclude that $\lambda(\ker(\varphi)) \leq \mathcal{F}_{n,m}^2$ whenever $\lambda \subsetneq 2^n$, hence $I \in \sum_{\lambda \subsetneq 2^n} \mathcal{F}^{(+)} \lambda(\ker(\varphi_{n,m}^2))$. Using the retraction $\mathcal{F} \rightarrow \mathcal{F}_{n,m}^2$ with kernel A , we conclude that I is contained in $(\mathcal{F}_{n,m}^2)^{(+)} \cdot \ker(\varphi_{n,m}^2)$.

Multidegree	3-variable monomials in the elements of $Q \setminus P$
(1, 1, 1)	$[xyz]$
(2, 1, 1)	$[xy][xz]$
(3, 1, 1)	$[x^2y][xz] \sim [x^2z][xy]$
(2, 2, 1)	$[x^2y][yz] \sim [xy^2][xz], [xyz][xy]^*$
(4, 1, 1) [*]	$[x^2y][x^2z]^*$
(3, 2, 1)	$[x^2y][xyz] \sim [x^2z][xy^2] \sim [xy]^2[xz]$
(2, 2, 2)	$[xyz]^2 \sim [x^2y][yz^2] \sim [x^2z][y^2z] \sim [xy^2][xz^2], [xy][xz][yz]^*$
(4, 2, 1) [*]	$[xy]^2[x^2z]^*, [x^2y][xy][xz]^*$
(3, 3, 1) [*]	$[xyz][xy]^2, [x^2y][xy][yz]^*, [xy^2][xy][xz]^*$
(3, 2, 2)	$[x^2y][xz][yz] \sim [xy^2][xz]^2 \sim [x^2z][xy][yz] \sim [xy]^2[xz^2], [xyz][xy][xz]^*,$
(5, 2, 1) [*]	$[x^2y]^2[xz]^*, [x^2y][x^2z][xy]^*$
(4, 3, 1) [*]	$[x^2y][x^2y][yz]^*, [x^2y][xyz][xy]^*, [x^2y][xy^2][xz]^*, [x^2z][xy^2][xy]^*, [xy]^3[xz]^*$
(4, 2, 2) [*]	$[x^2y][x^2z][yz]^*, [x^2y][xyz][xz]^*, [x^2y][xy][xz^2]^*,$ $[x^2z][xy^2][xz]^*, [x^2z][xyz][xy]^*, [xy]^2[xz]^2$
(3, 3, 2) [*]	$[x^2y][xyz][yz]^*, [x^2y][xy][yz^2]^*, [x^2y][xz][y^2z]^*, [x^2z][xy^2][yz]^*,$ $[x^2z][xy][y^2z]^*, [xy^2][xyz][xz]^*, [xy^2][xy][xz^2]^*, [xyz]^2[xy]^*, [xy]^2[xz][yz]^*$

Table 1: 3-variable monomials in $Q \setminus P$

Multidegree	Congruence	Relation	$\#\{S_3 \text{ translates}\}$	$\#\{S_4 \text{ translates}\}$
(3, 2, 0)	$[xy][x^2y] \equiv 0$	$r_{3,2}$	6	12
(3, 1, 1)	$[x^2y][xz] + [x^2z][xy] \equiv 0$	$r_{3,1,1}$	3	12
(2, 2, 1)	$[x^2y][yz] - [xy^2][xz] \equiv 0$	$r_{2,2,1}^{(1)}$	3	12
(2, 2, 1)	$[xy][xyz] \equiv 0$	$r_{2,2,1}^{(2)}$	3	12
(4, 2, 0)	$[x^2y]^2 \equiv 0$	$r_{4,2}$	6	12
(4, 1, 1)	$[x^2y][x^2z] \equiv 0$	$r_{4,1,1}$	3	12
(3, 3, 0)	$[xy]^3 + 3[x^2y][xy^2] \equiv 0$	$r_{3,3}$	3	6
(3, 2, 1)	$[x^2y][xyz] - [x^2z][xy^2] \equiv 0$	$r_{3,2,1}^{(1)}$	6	24
(3, 2, 1)	$[xy]^2[xz] + 3[x^2z][xy^2] \equiv 0$	$r_{3,2,1}^{(2)}$	6	24
(2, 2, 2)	$[xy][yz][zx] \equiv 0$	$r_{2,2,2}^{(1)}$	1	4
(2, 2, 2)	$[xyz]^2 - [xy^2][xz^2] \equiv 0$	$r_{2,2,2}^{(2)}$	3	12

Table 2: Relations in the case $m = 3$

Degree	Multidegree	Secondary generators
0	(0, 0, 0)	1
2	(1, 1, 0)	$[xy]$
3	(2, 1, 0)	$[x^2y]$
3	(1, 1, 1)	$[xyz]$
4	(2, 2, 0)	$[xy]^2$
4	(2, 1, 1)	$[xy][xz]$
5	(3, 1, 1)	$[x^2y][xz]$
5	(2, 2, 1)	$[x^2y][yz]$
6	(3, 3, 0)	$[x^2y][xy^2]$
6	(3, 2, 1)	$[x^2y][xyz]$
6	(2, 2, 2)	$[xyz]^2$
7	(3, 2, 2)	$[x^2y][xz][yz]$

Table 3: Secondary generators in the case $m = 3$

Multidegree	Congruence	Relation	# of S_4 -translates
(2, 1, 1, 1)	$[x^2y][zw] \equiv [xyz][xw] + [xyw][xz]$	$r_{2,1,1,1}$	12
(3, 1, 1, 1)	$[xy][xz][xw] \equiv -3[x^2y][xzw]$	$r_{3,1,1,1}$	12
(2, 2, 1, 1)	$[xy]^2[zw] \equiv 3[x^2z][y^2w] + 3[x^2w][y^2z] - 6[xyz][xyw]$	$r_{2,2,1,1}^{(1)}$	6
(2, 2, 1, 1)	$[xy][xz][yw] \equiv 3[x^2w][y^2z] - 3[xyz][xyw]$	$r_{2,2,1,1}^{(2)}$	12
(2, 2, 1, 1)	$[x^2y][yzw] \equiv [xyz][xyw]$	$r_{2,2,1,1}^{(3)}$	12

Table 4: 4-variable relations in the case $m = 4$

Multidegree	Secondary generators
(1, 1, 1, 1)	$[xy][zw], [xz][yw], [xw][yz]$
(2, 1, 1, 1)	$[xy][xzw], [xyw][xz], [xyz][xw]$
(3, 1, 1, 1)	$[x^2y][xzw]$
(2, 2, 1, 1)	$[x^2z][y^2w], [x^2w][y^2z], [xyz][xyw]$
(3, 2, 1, 1)	$[x^2y][xz][yw]$
(2, 2, 2, 1)	$[xy][xzw][yz], [xyw][xz][yz], [xy][xz][yzw]$
(3, 3, 1, 1)	$[x^2y][xyz][yw]$
(3, 2, 2, 1)	$[xyz]^2[xw]$
(2, 2, 2, 2)	$[x^2y][yz][zw^2], [x^2z][y^2w][zw], [x^2w][y^2z][zw]$

Table 5: 4-variable secondary generators

Multidegree	4-variable monomials in the elements of $Q \setminus P$
(1, 1, 1, 1)	$[xy][zw], [xz][yw], [xw][yz]$
(2, 1, 1, 1)	$[xy][xzw], [xz][xyw], [xw][xyz],$ $[x^2y][zw], [x^2z][yw], [x^2w][yz]$
(3, 1, 1, 1)	$[x^2y][xzw] \sim [x^2z][xyw] \sim [x^2w][xyz] \sim [xy][xz][xw]$
(2, 2, 1, 1)	$[xyz][xyw] \sim [x^2y][yzw] \sim [xy^2][xzw], [x^2z][y^2w], [x^2w][y^2z],$ $[xy]^2[zw], [xy][xz][yw], [xy][xw][yz]$
(4, 1, 1, 1) [*]	$[x^2y][xz][xw]^*, [x^2z][xy][xw]^*, [x^2w][xy][xz]^*$
(3, 2, 1, 1)	$[x^2y][xz][yw] \sim [x^2y][xw][yz] \sim [x^2z][xy][yw] \sim$ $\sim [x^2w][xy][yz] \sim [xy^2][xz][xw] \sim [xy]^2[xzw],$ $[x^2y][xy][zw]^*, [xyz][xy][xw]^*, [xyw][xy][xz]^*$
(2, 2, 2, 1)	$[xy][yz][xzw] \sim [x^2z][yw][yz] \sim [xz^2][xy][yw],$ $[xz][yz][xyw] \sim [x^2y][yz][zw] \sim [xy^2][xz][zw],$ $[xz][xy][yzw] \sim [xz][xw][y^2z] \sim [xy][xw][yz^2],$ $[x^2w][yz]^2, [xy]^2[z^2w], [xz]^2[y^2w],$ $[xyz][xy][zw]^*, [xyz][xz][yw]^*, [xyz][xw][yz]^*$
(5, 1, 1, 1) [*]	$[x^2y][x^2z][xw]^*, [x^2y][x^2w][xz]^*, [x^2z][x^2w][xy]^*$
(4, 2, 1, 1) [*]	$[x^2y]^2[zw]^*, [x^2y][x^2z][yw]^*, [x^2y][x^2w][yz]^*, [x^2y][xyz][xw]^*,$ $[x^2y][xyw][xz]^*, [x^2y][xy][xzw]^*, [x^2z][xy^2][xw]^*, [x^2z][xyw][xy]^*,$ $[x^2w][xy^2][xz]^*, [x^2w][xyz][xy]^*, [xy]^2[xz][xw]^*$
(3, 3, 1, 1)	$[x^2y][xyz][yw] \sim [x^2y][y^2z][xw] \sim [y^2z][xy][x^2w] \sim [xy^2][yz][x^2w] \sim$ $\sim [x^2y][yz][xyw] \sim [xy^2][xz][xyw] \sim [x^2y][xz][y^2w] \sim [x^2z][xy][y^2w] \sim$ $\sim [x^2z][xy^2][yw] \sim [xy^2][xyz][xw] \sim [xy]^2[xz][yw] \sim [xy]^2[yz][xw] \sim$ $\sim [x^2y][xy^2][zw] \sim [xy]^3[zw],$ $[xyz][xy][xyw]^*, [x^2y][xy][yzw]^*, [xy^2][xy][xzw]^*$
(3, 2, 2, 1)	$[xyz]^2[xw] \sim [x^2y][xyz][zw] \sim [x^2z][xyz][yw] \sim [x^2z][xy^2][zw] \sim$ $\sim [x^2y][xz^2][yw] \sim [x^2y][yz^2][xw] \sim [x^2z][y^2z][xw] \sim [xy^2][xz^2][xw] \sim$ $\sim [y^2z][xz][x^2w] \sim [yz^2][xy][x^2w] \sim [xy]^2[xz][zw] \sim [xz]^2[xy][yw],$ $[xy][xz][yz][xw]^*, [x^2y][xz][yzw]^*, [x^2z][xy][yzw]^*, [x^2y][yz][xzw]^*,$ $[x^2z][yz][xyw]^*, [x^2y][xy][z^2w]^*, [x^2z][xz][y^2w]^*, [xy^2][xz][xzw]^*,$ $[xyz][xy][xzw]^*, [xz^2][xy][xyw]^*, [xyz][xz][xyw]^*, [xyz][yz][x^2w]^*$
(2, 2, 2, 2)	$[xz][yz][xw][yw] \sim [x^2y][yz][zw^2] \sim [xy^2][xz][zw^2] \sim [x^2y][yw][z^2w] \sim [xy^2][xw][z^2w],$ $[xy][yz][xw][zw] \sim [x^2z][y^2w][zw] \sim [x^2z][yz][yw^2] \sim [xz^2][xy][yw^2] \sim [xz^2][xw][y^2w],$ $[xy][xz][yw][zw] \sim [x^2w][y^2z][zw] \sim [x^2w][yz^2][yw] \sim [xz][xw^2][y^2z] \sim [yz^2][xy][xw^2],$ $[xyz][xy][zw^2]^*, [xyz][xz][yw^2]^*, [xyz][yz][xw^2]^*, [x^2y][zw][yzw]^*,$ $[x^2z][yw][yzw]^*, [xy][xzw][yzw]^*, [xyw][xz][yzw]^*, [xyz][xw][yzw]^*$ $[xy^2][zw][xzw]^*, [y^2z][xw][xzw]^*, [xyw][yz][xzw]^*, [xyz][yw][xzw]^*,$ $[xz^2][yw][xyw]^*, [yz^2][xw][xyw]^*, [xyz][zw][xyw]^*, [xyw][xy][z^2w]^*,$ $[xzw][xz][y^2w]^*, [yzw][yz][x^2w]^*,$ $[xy]^2[zw]^2, [xz]^2[yw]^2, [yz]^2[xw]^2$

Table 6: 4-variable monomials in $Q \setminus P$

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